Spectral Analysis on Riemannian Manifolds

Spherical Harmonic Decomposition of Global Financial Covariance Structures

WP-2025-004-QUANT

From Wiener-Khinchin Theorem to High-Dimensional Alpha Generation

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Date:
February 10, 2026
Location:
Lugano, Switzerland

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1 Abstract

This paper explores the rigorous application of **Harmonic Analysis** to stochastic financial time-series. Moving beyond elementary signal processing, we establish the measure-theoretic foundations of spectral density estimation in L^2 Hilbert spaces.

We extend the analysis from the Euclidean domain $\mathbb{R}^n$ to the spherical domain $\mathbb{S}^2$, introducing **Spherical Harmonics** (Y_l^m) as a basis for modeling global macro-financial correlations and diffusive processes on curved manifolds. Finally, we propose a novel "Spectral Arbitrage" framework, utilizing the phase discrepancies between low-frequency trend components and high-frequency noise to isolate alpha in non-stationary regimes.

2 Mathematical Foundations: L^2 Spaces and Hilbert Theory

Financial time-series are technically realizations of stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P)$. To apply Fourier analysis rigorously, we must operate within the Hilbert space of square-integrable functions.

2.1 The Space $L^2(\mathbb{T})$

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the torus. We define the space of finite energy signals as:

$$L^2(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{C} \;\middle|\; \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \right\} \tag{1}$$

This space is equipped with the inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx \tag{2}$$

2.2 Fourier Series Convergence

For any $f \in L^2(\mathbb{T})$, the Fourier coefficients are defined as:

$$\hat{f}(n) = \langle f, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z} \tag{3}$$

The crucial result for financial approximation is **Plancherel's Theorem**, which states that the energy in the time domain equals the energy in the frequency domain (Isometry):

$$\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \tag{4}$$

In finance, this implies that the variance of the return series is the sum of the variances of its spectral components.

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3 Stochastic Calculus in the Frequency Domain

Financial data is rarely deterministic. We must bridge Fourier analysis with Stochastic Calculus.

3.1 The Wiener-Khinchin Theorem

For a wide-sense stationary (WSS) process X_t, the Autocorrelation Function (ACF) is defined as $R_X(\tau) = \mathbb{E}[X_t X^*_{t+\tau}]$.

Theorem 1 (Wiener-Khinchin)**.** *The Power Spectral Density (PSD), denoted $S_X(\omega)$, is the Fourier Transform of the Autocorrelation Function:*

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} d\tau \tag{5}$$

Financial Implication: By analyzing $S_X(\omega)$, Equilibrium algorithms can detect hidden periodicities (e.g., intra-day liquidity cycles or seasonal macro-trends) that are invisible in the time domain variance.

3.2 Application: The Carr-Madan Formula for Option Pricing

Pricing derivatives often requires solving convolution integrals. The Fast Fourier Transform (FFT) reduces complexity from $O(N^2)$ to $O(N \log N)$. Let $\psi_T(u)$ be the characteristic function of the log-price density. The call option price $C(k)$ is:

$$C(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \frac{e^{-rT} \psi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} dv \tag{6}$$

This allows for real-time calibration of Levy processes (e.g., Variance Gamma) to volatility surfaces.

4 Spherical Harmonics: Analysis on the 2-Sphere $\mathbb{S}^2$

While Fourier series apply to periodic signals on a line (or circle), global financial data (e.g., supply chain flows, geopolitical risk heatmaps) naturally resides on the surface of a sphere. Furthermore, correlation matrices of assets can be mapped to hyperspheres.

4.1 The Spherical Laplacian

The Laplacian operator Δ in spherical coordinates (θ, ϕ) is:

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{7}$$

We seek eigenfunctions Y such that $\Delta Y = \lambda Y$.

4.2 Legendre Polynomials and Spherical Harmonics

The solutions are the **Spherical Harmonics** $Y_l^m(\theta, \phi)$, defined as:

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \tag{8}$$

where P_l^m are the Associated Legendre Polynomials. These functions form an orthonormal basis for $L^2(\mathbb{S}^2)$.

4.3 Spherical Decomposition of Financial Fields

Any square-integrable scalar field $F(\theta, \phi)$ (e.g., a global inflation index mapped to coordinates) can be decomposed as:

$$F(\theta, \phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_l^m(\theta, \phi) \tag{9}$$

The coefficients a_{lm} represent the "power" of the signal at different angular frequencies.

- $l = 0$: Global average (Monopole).
- $l = 1$: North-South / East-West gradients (Dipole).
- $l = 2$: Quadrupole moments (Complex regional disparities).

5 Advanced Proposal: The "Spectral Arbitrage" Strategy

We propose a proprietary trading framework that exploits the **Phase-Lag** between spectral components of correlated assets.

5.1 Methodology

1. **Decomposition:** Decompose the price series P_t of two cointegrated assets A and B into spectral bands $P_t^{(k)}$ using the Discrete Wavelet Transform (DWT) or Short-Time Fourier Transform (STFT).

$$P_t = \sum_k P_t^{(k)} + \epsilon_t \tag{10}$$

2. **Cross-Spectral Analysis:** Compute the Cross-Spectral Density $S_{AB}(\omega)$. 3. **Phase Extraction:** Calculate the Phase Spectrum $\phi_{AB}(\omega) = \arg(S_{AB}(\omega))$. 4. **Signal Generation:**

$$\text{Signal}_t = \int_{\omega_{low}}^{\omega_{high}} |\phi_{AB}(\omega)| \cdot \text{Coherence}_{AB}(\omega) d\omega \tag{11}$$

5.2 Practical Application

If $\phi_{AB}(\omega_0) > 0$, asset A leads asset B at frequency ω_0. The strategy goes **Long B / Short A** to capture the mean reversion of the phase lag. This is structurally different from time-domain pairs trading (Ornstein-Uhlenbeck) as it isolates specific cycle latencies.

6 Conclusion and Future Research

The transition from Time Domain to Frequency Domain via Fourier and Spherical analysis offers a higher-dimensional perspective on market dynamics.

By modeling global data on $\mathbb{S}^2$ using Spherical Harmonics, Equilibrium can capture geometric dependencies that linear correlation matrices miss. Furthermore, the rigorous application of the **Wiener-Khinchin theorem** allows for the construction of "Spectral Alphas" that are orthogonal to traditional momentum or mean-reversion factors.

Future research will focus on **Hyper-Spherical Harmonics** on $\mathbb{S}^n$ to model the latent space of the entire S&P 500 covariance matrix ($n = 500$), applying heat kernel diffusion methods to predict correlation breakdowns.

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